This homework introduces you to some interesting, special matrices, giving some feel for what matrices do to vectors. General guidelines: Read through each complete problem carefully before attempting any parts. Feel free to collaborate in groups of size 2-3, but always note the names of your collaborators on your submitted homework. For graphs: clearly label your axes and use good color and symbol choices. Print out your matlab code (in the form of a script file). For derivations you’re asked to do ‘by hand’ (in other words, analytically, using paper and pencil) feel free to turn in handwritten or typed-out work.

1) Unit vectors, angle between vectors, and vector projection.
   a. Derive the unit vectors $e_u, e_v$ in the direction of $u = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
   b. Derive the inner angle between $u$ and $v$. Show your work.
   c. Express $u$ with a different set of basis vectors. In a. above, we wrote $u$ in the standard basis of $\hat{e}_1 = (1, 0), \hat{e}_2 = (0, 1)$ as $u = 3\hat{e}_1 + 1\hat{e}_2$. Thus, the coefficients or projections of $u$ onto that standard basis are 3, 1, respectively. The new basis is $\hat{e}'_1 = \frac{1}{\sqrt{2}}(1, 1), \hat{e}'_2 = (-1, 1)$. Is this an orthogonal basis? Are the basis vectors normalized (unit length)? First draw by hand the old basis, the new basis, and the vector $u$. Also by hand on the same plot, show how to project $u$ onto the old and new bases. Derive the coefficients of $u, v$ in the new basis. Show your work.

2) Some special matrices, plotting vectors in Matlab.
   a. Consider the $2 \times 2$ matrix $M(\theta)$, given by
      $$
      M(\theta) = \begin{pmatrix}
      \cos(\theta) & -\sin(\theta) \\
      \sin(\theta) & \cos(\theta)
      \end{pmatrix}.
      $$
      for some choice of $\theta$ (you choose). In Matlab, multiply $M$ into the vector $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Plot $v$ and $v' = Mv$ on the same plot. (Hint: use plotv) Next generate $v'' = Mv'$ (verify that and plot that on top of the previous plot. (Note that
\( \mathbf{v}', \mathbf{v}'' \) do not refer to transposes of \( \mathbf{v} \) here, even though primes denote vector and matrix transposition in Matlab; here they are simply the names of different vectors.) Repeat for various different choices of \( \theta \). State precisely how \( \mathbf{v}, \mathbf{v}' \) and \( \mathbf{v}'' \) differ and in what ways they remain the same. Next, compute the product \( M\mathbf{v} \) by hand, for general \( \theta \). What can you conclude about the effect of \( M(\theta) \) on vectors? What might you want to call the matrix \( M \)? Finally, compute \( (M(\theta))^2 \) in terms of the variable \( \theta \), and rewrite each element as just a cosine or a sine, using the double-angle/half-angle formulae for sines and cosines. What does applying \( M \) repeatedly do to a vector?

b. Consider another \( 2 \times 2 \) matrix \( R \), given by

\[
R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Right-multiply \( R \) by hand (analytically) with an arbitrary vector \( \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \), to generate \( \mathbf{v}' = R\mathbf{v} \). Predict what will happen if you repeat the process, and multiply \( \mathbf{v}' \) by \( R \)? Do not do this in Matlab. Instead, use the observation that \( \mathbf{v}'' = R\mathbf{v}' = R^2 \mathbf{v} \), and by hand (not in Matlab), compute \( R^2 \).

Summarize the operation that \( R \) performs on vectors. What might you want to call it?

c. Consider the \( 2 \times 2 \) matrix \( A \), given by

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

and consider the column vector \( \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \). By hand, compute \( \mathbf{y} = A\mathbf{v} \) and \( \mathbf{z}^T = \mathbf{v}^T A \). Is \( \mathbf{y} = \mathbf{z} \)? If not, what condition on \( A \) will guarantee that they are? This is the definition of a symmetric matrix.

3) Orthogonal matrices

a. Length-preserving matrices: Derive the conditions that an arbitrary-dimension square matrix \( A \) must obey, to preserve the norms of all vectors it acts on. (Hint: recall that the squared norm of a real-valued column vector \( \mathbf{x} \) is given by \( \mathbf{x}^T \mathbf{x} \).) This is the definition of an orthogonal matrix.

b. Are the matrices \( M(\theta) \) and \( R \) above orthogonal matrices?
c. Determinant: As we have already seen, an important scalar quantity associated with a square matrix is its determinant. Compute the determinant of the matrices $M(\theta)$ and $R$ above (without substituting in a specific value of $\theta$). Orthogonal matrices always have determinant equal to $\pm 1$.

d. Here’s another orthogonal matrix:

$$P = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix}$$

Verify by hand that it’s orthogonal, show your work. Figure out what it does, and explain.